Scalar Field Cosmological Models With Hard Potential Walls

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The global behavior of scalar field cosmological models with very hard potential walls is investigated via the simple example of an exponentially steep potential well. It is found that the solutions exhibit a non-trivial oscillatory behavior in the approach to an initial space-time singularity. This behavior can be interpreted as being due to the inability of the scalar field to negotiate the walls of the steep potential well.

I. INTRODUCTION

According to the inflationary universe scenario [1–3] the matter content in the very early universe can be modeled by a scalar field ϕ with a non-negative self interaction $V(\phi)$. Inflation may be understood physically in terms of the stress energy tensor of the scalar field which is equivalent to a perfect fluid with energy density ρ and pressure p given by

$$\rho = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + V(\phi) \qquad p = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi). \tag{1}$$

When V becomes large compared to the gradient of the scalar field, the pressure becomes large and negative and the fluid satisfies the approximate equation of state $\rho = -p$ leading to a rapid exponential expansion of space-time. This rapid expansion is associated with the fact that gravity becomes repulsive in the presence of large negative pressures. Such equations of state have traditionally been forbidden in general relativistic matter fields by the strong energy condition which, for a perfect fluid, is equivalent to the following familiar constraint on the pressure and energy [4]:

$$\rho + 3p > 0.$$

Thus, for a scalar field to obey the strong energy condition we would require that

$$-\partial_\mu\phi\partial^\mu\phi - V(\phi) > 0.$$

The fact that scalar fields naturally violate the above inequality means that the singularity theorems [4] can not be strictly applied when such fields are present in the early universe. Indeed, it is relatively simple to construct exact singularity free scalar field cosmologies (eg [5]). Other exact scalar field solutions are known which possess singularities, but have no particle horizons [6], indicating that scalar field cosmologies can exhibit a diverse range of behavior.

It turns out however, that all of the known singularity free and horizon free solutions are unstable and it can be demonstrated that, provided the potential diverges slower than $\exp(\sqrt{6}\phi)$ as $\phi \to \pm\infty$, almost all spatially flat Friedmann Robertson Walker (FRW) cosmologies have initial singularities. Furthermore, they have asymptotic equation of state $\rho = p$ and may be asymptotically approximated by the general solution for the massless scalar field [7].

However, since we have no direct emperical data with which to determine the detailed nature of physical fields at energy densities comparable with those of the early universe, we should not rule out the possibility that scalar fields present in the early universe might have much steeper potentials.

It is therefore natural to ask whether singularity free or horizon free cosmologies can arise as a result of very steep potentials. In particular, we might expect that a very steep potential well could inhibit the divergence of the scalar field, thereby slowing down the gravitational collapse and resulting in singularity or particle horizon avoidance.

Indeed, it can easily be demonstrated that for very steep potentials V can not be neglected asymptotically in the past as can be done with less steep potentials. This can be seen as follows:

For simplicity, we confine attention to the FRW line element

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3}(dx^i)^2. \tag{2}$$

We define the expansion $K(t)$ as the function

$$K = 3\frac{\dot{a}}{a}, \tag{3}$$

where the "dot" indicates differentiation with respect to time t. K may be interpreted physically as the rate of expansion of the spatial volume element $v = a^3$.

The evolution equations governing the interaction of a general scalar field with the above metric are [3]:

$$\ddot{\phi} = -K\dot{\phi} - V'(\phi) \tag{4}$$
$$\dot{K} = -\tfrac{3}{2}\dot{\phi}^2$$

$$K^2 = 3V(\phi) + \frac{3}{2}\dot{\phi}^2. \tag{5}$$

The general solution for the massless scalar field is obtained by setting $V = 0$ in the above and solving to obtain.

$$K = \frac{1}{t} \qquad \phi = \pm\sqrt{\tfrac{2}{3}} \ln \frac{t}{c} \qquad (6)$$

Let us now use (6) as an approximation to the solution for arbitrary V in the limit $t \to 0$. Substituting (6) into (5) we obtain

$$\frac{1}{t^2} = \frac{1}{t^2} + V\left(\pm\sqrt{\tfrac{2}{3}} \ln \frac{t}{c}\right) + h$$

where, h indicates terms of higher order in t. For this expression to be consistent the following must be true:

$$\lim_{t \to 0} t^2 V\left(\pm\sqrt{\tfrac{2}{3}} \ln \frac{t}{c}\right) = 0.$$

i.e.

$$\lim_{\phi \to \pm\infty} e^{-\sqrt{6}|\phi|} V(\phi) = 0.$$

In other words, in order for the massless scalar field to represent a consistent asymptotic approximation of a scalar field model it is necessary that the potential diverge slower than $e^{-\sqrt{6}|\phi|}$. Exponential potentials therefore seem to accommodate a kind of transition between soft wall potentials and some other qualitative regime.

In this paper we examine the behavior of very steep potentials by means of a simple and interesting example.

II. THE MODEL AND DYNAMICAL EQUATIONS.

The simplest example of a potential which is exponentially steep at both infinity and negative infinity is an exponential well of the form.

$$V(\phi) = a e^{\lambda\phi} + b e^{-\mu\phi}.$$

where a, b, λ and μ are positive constants. In the subsequent analysis we will confine attention to the case where $a = b = 1$ and $\lambda = \mu$ so that

$$V(\phi) = e^{\lambda\phi} + e^{-\lambda\phi}. \qquad (7)$$

By increasing λ we shall be able to investigate how the qualitative behavior of the system changes as the potential well becomes increasingly steep. The analysis for the more general potential is similar and the conclusions are essentially the same.

We are interested only in expanding solutions of (4) so we may replace K and $\dot\phi$ with the new set of coordinates

$$x = \frac{1}{K} \qquad y = \sqrt{\tfrac{3}{2}} \frac{\dot\phi}{K} \qquad (8)$$

and introduce a new time coordinate

$$\tau = \ln v(t) + \tau_0 \qquad (9)$$

where $v = a^3$ is the spatial volume element and τ_0 is some constant. τ is well defined since v is strictly increasing. Furthermore, v goes to zero if and only if K goes to infinity [4] so (9) implies that $K \to \infty$ as $\tau \to -\infty$. Differentiating (9) we find (recalling that $K = \dot{v}/v$)

$$\frac{d}{dt} = K\frac{d}{d\tau}. \qquad (10)$$

In terms of these coordinates the field equations (4) with the potential (7) may be written as a dynamical system:

$$\frac{dx}{d\tau} = y^2 x$$
$$\frac{dy}{d\tau} = -y - 3\alpha x^2 (e^{\sqrt{6}\alpha\phi} - e^{-\sqrt{6}\alpha\phi}) + y^3 \qquad (11)$$
$$\frac{d\phi}{d\tau} = y$$

where $\alpha = \frac{\lambda}{\sqrt{6}}$. The "constraint" equation (5) becomes

$$y^2 + 3x^2 (e^{\sqrt{6}\alpha\phi} + e^{-\sqrt{6}\alpha\phi}) = 1. \qquad (12)$$

We can simplify the equations by defining the variables

$$p = \sqrt{3} e^{-\sqrt{\frac{3}{2}}\alpha\phi} x \qquad q = y. \qquad (13)$$

then the constraint equation may be written

$$p^2 + q^2 = 1 - p^2 e^{2\sqrt{6}\alpha\phi}. \qquad (14)$$

Substituting (13) and (14) into (11) we obtain

$$\frac{dp}{d\tau} = -\alpha pq + pq^2$$
$$\frac{dq}{d\tau} = q^3 + \alpha q^2 - q - \alpha + 2\alpha p^2. \qquad (15)$$

These equations constitute a 2-dimensional dynamical system on the p-q plane. We may define the physical phase space Ω, according to (14), as the set $\Omega = \{(p,q) : p^2 + q^2 < 1, p > 0\}$. In other words, all physical trajectories lie on the interior of the unit disc to the right of the q axis. The unphysical boundary $\partial\Omega$ is a closed curve consisting of the union of the smooth arc $\partial\Omega_1 = \{(p,q) : p^2 + q^2 = 1, p > 0\}$ and the line segment $\partial\Omega_2 = \{(p,q) : p = 0, |q| \leq 1\}$. The asymmetric appearance of $\partial\Omega$ is a consequence of the positive exponential term in the definition of the coordinate p and is not a physical property of the system itself. In fact it should be pointed out that $\partial\Omega_1$ maps onto $\partial\Omega_2$ under the transformation $(\phi, \dot\phi) \mapsto -(\phi, \dot\phi)$. $\partial\Omega$ corresponds to the infinity of the expansion K. In order to see this define the function

$$H(p,q) = p^2(1 - p^2 - q^2) \qquad (16)$$

Clearly H is strictly positive everywhere on Ω and vanishes identically on $\partial\Omega$. From (14) and (13) we see that

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$$H = p^4 e^{2\sqrt{6}\alpha\phi} = 9x^4. \tag{17}$$

Thus $\partial\Omega$ is just the set of all points for which $x = 0$ which is by definition the infinity of K. Evaluating the directional derivative of H along the flow using (15) we find

$$\frac{dH}{d\tau} = 4Hq^2 \tag{18}$$

which is non-negative everywhere on the interior of Ω. In fact the derivative of H with respect to τ is strictly positive everywhere on the interior of Ω except where $q = 0$. Observe also that $\dfrac{dH}{d\tau} = 0$ on $\partial\Omega$ indicating that the boundary is an invariant manifold (tangent to the flow). No physical trajectories can therefore cross $\partial\Omega$ into the unphysical domain beyond. Since Ω is compact all trajectories must possess an past-limit set which is invariant under the flow. Since H is monotonic, any limit point must have $\dfrac{dH}{d\tau} = 0$. Therefore, all limit sets must be subsets of either $\partial\Omega$ or the line $q = 0$. From (15) the only invariant subset of $q = 0$ is the equilibrium point $\mathbf{p}_d = (\frac{1}{\sqrt{2}}, 0)^2$. However this point is a local maximum of H as is easily verified by evaluating its gradient. Since H is monotonic increasing, no solutions can be past asymptotic to $\mathbf{p}_d$ other than the steady state solution on $\mathbf{p}_d$ itself. It follows that the past-limit sets of all other solutions lie on the boundary $\partial\Omega$.

By the time reverse of the above argument it is clear that all solutions, with the exception of those unphysical solutions lying on $\partial\Omega$ are future asymptotic to $\mathbf{p}_d$.

The future asymptotic set $\mathbf{p}_d$ represents a vacuum de Sitter space-time with constant expansion $K = \sqrt{6}$ and ϕ identically zero. This is consistent with what we would expect for a scalar field cosmology with non-zero vacuum energy.

To summarize, it has been established that (almost) all solutions originate near $\partial\Omega$ and subsequently evolve towards the global attractor $\mathbf{p}_d$ (which represents de Sitter space) as $t \to \infty$. Let us now examine the behavior of the system on and near $\partial\Omega$ in more detail.

III. THE BEHAVIOR CLOSE TO $\partial\Omega$.

There are a maximum 4 equilibrium points of (15) lying on $\partial\Omega$. These are $(\sqrt{(1-\alpha^2)}, \alpha)$, $(0,1)$, $(0,-1)$ and $(0,-\alpha)$, which we label $\mathbf{p}_1$, $\mathbf{p}_+$, $\mathbf{p}_-$ and $\mathbf{p}_2$ respectively. Observe that $\mathbf{p}_1$ and $\mathbf{p}_2$ only exist as distinct equilibrium points on $\partial\Omega$ when $\alpha < 1$. For values of α greater than or equal to one, $\mathbf{p}_\pm$ are the only equilibrium points. In

order to investigate the behavior close to $\mathbf{p}_\pm$ equations (15) and (16) can be linearized about these points. The linearized system is:

$$\frac{dp}{d\tau} = (1 \mp \alpha)p \tag{19}$$

$$\frac{dq}{d\tau} = 2(1 \pm \alpha)(q \mp 1) \tag{20}$$

The solution to the linear system is

$$p = p_0 e^{(1\mp\alpha)\tau} \qquad q = \pm 1 \mp \delta_0 e^{2(1\pm\alpha)\tau} \tag{21}$$

where p_0 and δ_0 are positive constants.

A. The Flow For $\alpha < 1$.

When $\alpha < 1$ both exponential terms have positive coefficients indicating that $\mathbf{p}_\pm$ are sources of the linear system and therefore, by the Hartman-Grobman Theorem [8,9], of the non-linear system also. Since the unphysical solutions on $\partial\Omega$ move away from $\mathbf{p}_\pm$ they must asymptotically approach $\mathbf{p}_1$ and $\mathbf{p}_2$ (in the forwards time sense). However all *physical* solutions (ie, those on the interior of Ω) asymptotically approach $\mathbf{p}_d$ in the future. $\mathbf{p}_1$ and $\mathbf{p}_2$ must therefore be unstable saddles. This can be seen more clearly from inspection of the geometry of the flow as illustrated in Fig. 1.
(An alternative way to verify that $\mathbf{p}_1$ and $\mathbf{p}_2$ are saddles is by linearizing (15) about these points but this exercise shall be left to the reader who remains unconvinced).
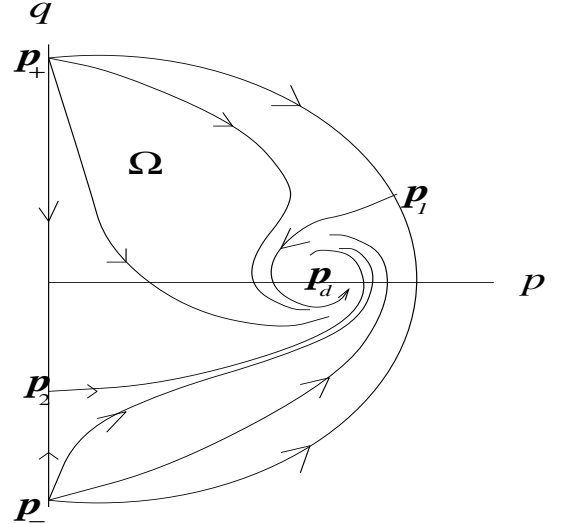

FIG. 1. Sketch of Ω showing all possible equilibrium points, direction of the flow on the boundary, and some typical trajectories. Trajectories on the boundary are future asymptotic to $\mathbf{p}_1$ and $\mathbf{p}_2$ but all trajectories on the interior approach $\mathbf{p}_d$.

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[2]In order to avoid confusion points on the (p,q) plane will be labeled in bold print.

With the exception of the 2 solutions past asymptotic to $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively, and the steady state solution lying on $\mathbf{p}_d$, all physical trajectories must asymptotically approach either $\mathbf{p}_+$ or $\mathbf{p}_-$ as $\tau \to -\infty$, as indicated in Fig. 1.

Using the fact that $1 - q^2$ decays exponentially to zero as $\tau \to -\infty$ we may integrate (18) to obtain a first order expression for H:

$$H = H_0 e^{4\tau}$$

and hence

$$x = x_0 e^{\tau}.$$

Using (10) we obtain a first order expression for t:

$$t = x_0 e^{\tau}.$$

Observe that $t \to 0$ as $\tau \to -\infty$ indicating that $\mathbf{p}_\pm$ correspond to space-time singularities. Using the definitions of x and q we thus obtain the asymptotic solution for K and $\dot{\phi}$ in the neighbourhood of $t = 0$:

$$K = \frac{1}{t} \qquad \dot{\phi} = \pm\sqrt{\frac{2}{3}}\frac{1}{t} \qquad (22)$$

and upon integration of $\dot{\phi}$

$$\phi = \pm\sqrt{\frac{2}{3}}\ln\frac{t}{c}. \qquad (23)$$

This is, of course, the general solution for the massless scalar field. We thus conclude that when $\alpha < 1$, ie $\lambda < \sqrt{6}$, the potential V is not dynamically significant near the singularity. This is precisely the behavior we would have expected from the simple calculation in Section 1 for potentials that go to infinity slower than $\exp(\sqrt{6}\phi)$. In fact the topological structure of solution space is typical of inflationary models. All s solutions emerge from the sources $\mathbf{p}_\pm$ except for the seperatrices I_1 or I_2 which we define as the unique solutions past asymptotic to the saddles $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively. We may infer that these solutions characterize the inflation in the system. (It can be easily verified that the behavior close to $\mathbf{p}_1$ and $\mathbf{p}_2$ is approximated by the exact power law cosmologies [6]. However this behavior only corrosponds to inflation when $\alpha^2 < \frac{1}{3}$ since this condition ensures that the square of the q coordinate at $\mathbf{p}_+$ and $\mathbf{p}_-$ is less than $\frac{1}{3}$ which is necessary and sufficient for violation of the strong energy condition.)

B. The Flow For $\alpha \geq 1$

Let us now consider what happens when $\alpha \geq 1$. As we demonstrated in Section 1, the potential in this case becomes too steep for the solution with $V = 0$ to be consistent as an asymptotic solution. In more physical terms, the gravitational expansion is unable to dominate the scalar field self interaction in the initial expansion phase of the universe. For expediency, we shall assume in what follows that $\alpha > 1$. The special case $\alpha = 1$ differs in the details of analysis, but the qualitative features of the solutions turn out to be essentially the same.

The only equilibrium points possessed by the system when $\alpha > 1$ are $\mathbf{p}_+$, $\mathbf{p}_-$ and the sink $\mathbf{p}_\infty$. Inspection of (21) reveals that $\mathbf{p}_\pm$ become hyperbolic saddles in this regime.

The only solution past asymptotic to $\mathbf{p}_-$ is the unphysical trajectory γ_1 which emerges from $\mathbf{p}_-$ at $\tau = -\infty$ and proceeds anticlockwise along $\partial\Omega_1$ (the unit circle), reaching $\mathbf{p}_+$ at $\tau = \infty$. Similarly, the only solution which originates at $\mathbf{p}_+$ is the unphysical trajectory γ_2 which emerges from $\mathbf{p}_+$ at $\tau = -\infty$ and proceeds along $\partial\Omega_2$ (the q-axis), approaching $\mathbf{p}_2$ asymptotically as $\tau \to \infty$. The union of γ_1, γ_2 is a closed heteroclinic cycle γ_L on $\partial\Omega$ which forms a limit cycle for trajectories on the interior of Ω. That is, as $\eta \to -\infty$ a typical trajectory will approach $\partial\Omega$, spiraling clockwise (in the reverse time sense) an infinite number of times. The solution will asymptotically approach the non-physical solutions γ_1 and γ_2 but unlike these will always avoid the equilibrium points $\mathbf{p}_+$ and $\mathbf{p}_-$ and will continue to spiral around $\partial\Omega$ *ad infinitum*. Fig 2.
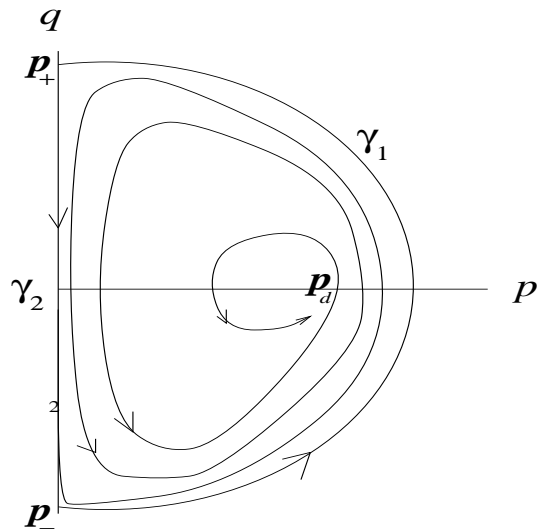

FIG. 2. Sketch of Ω for $\alpha > 1$ showing the solutions on the boundary and a typical solution on the interior. The points $\mathbf{p}_\pm$ are saddles and therefore unstable, but solutions must approach the boundary so the union of solutions on $\partial\Omega$ forms a limit cycle.

In order to interpret these results recall firstly that, from (16) and (18), H monotonically decreases to 0 as $\tau \to -\infty$ and hence the expansion K diverges monotonically to infinity. The asymptotic periodic behavior of the trajectories must therefore represent oscillations of the scalar field ϕ. Inspection of (14) indicates that $\phi = -\infty$ on the semi-circle $\partial\Omega_1$. It follows from symmetry that

$\phi = \infty$ on $\partial\Omega_2$. As the gravitational expansion diverges to infinity (and correspondingly the scale parameter a tends to 0) the scalar field oscillates about $\phi = 0$ with the amplitude of oscillation increasing for each successive cycle, asymptotically diverging to infinity.

No asymptotic equation of state is satisfied by the scalar field. In the "corners" of Ω, close to the points $p_\pm$, the equation of state is approximately $\rho = p$ since $\dot{\phi}^2 \simeq \frac{2}{3}K^2$ which implies, via (5) that $\dot{\phi}^2 >> V(\phi)$. This corresponds to the period where the scalar field is close to the bottom of the potential well.

Close to the q-axis, on the other hand, the equation of state is approximately $\rho = -p$ since $\dot{\phi} \simeq 0$ whilst $V(\phi) \geq 2$. This corresponds to the scalar field reaching the top of its roll before accelerating down the potential well again. The asymptotic behavior of the space-time may therefore be characterized by an infinite sequence of "stiff" phases, punctuated by de Sitter phases. During the de Sitter phase the matter violates the strong energy condition.

IV. THE EXISTENCE OF A SINGULARITY AND PARTICLE HORIZON

Since an infinite number of physical cycles of the scalar field occur for the case $\alpha > 1$, it might reasonably be expected that an infinite interval of proper time must also elapse since the oscillations of the scalar field could be used as a natural physical clock which would measure an infinite time interval to the past of any space-time point. If this were the case then the space-time would be non-singular for all physical solutions. It turns out however, that the oscillations pile up on each other and a space-time singularity is indeed reached after a finite time interval.

In order to show this we must estimate the proper time that elapses to the past of some arbitrary initial point $\mathbf{p}_0$, on the trajectory $\psi_{\mathbf{p}_0}(\tau)$. Since all trajectories are asymptotically tangent to the boundary $\partial\Omega$ it will be sufficient to restrict attention to initial points lying in the set $\Sigma_0 = \{(p,q) : q = 1 - \epsilon,\ 0 < p \leq \epsilon\}$ where $\epsilon > 0$ may be chosen arbitrarily small.

Before proving that space-time singularities and particle horizons exist for all solutions it will be convenient to prove the following lemma;

Lemma 1 *For all $0 < n < 1$ there exist positive numbers ϵ and p_m such that if $x_0 = (p_0, 1-\epsilon)$ is in Σ_0 and $p_0 < p_m$ then*

$$H(\psi_{\mathbf{p}_0}(\tau)) < H_0 e^{4n\tau} \qquad (24)$$

to the past of $\mathbf{p}_0$, where $\psi_{x_0}(\tau)$ is the unique trajectory of (15) passing through $\mathbf{p}_0$ with $\psi_{\mathbf{p}_0}(0) = \mathbf{p}_0$.

Proof: Integrating (18) backwards in time from $\tau = 0$ to some earlier time τ_f we obtain

$$H(\tau_f) = H_0 \exp\left[-4\int_{\tau_f}^0 q^2 d\tau\right]. \qquad (25)$$

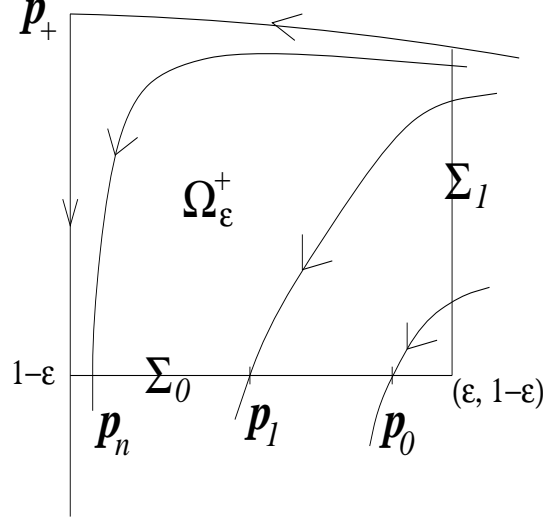Let $\Sigma_1 = \{(p,q) \in \Omega : p = \epsilon,\ 1 - q \leq \epsilon\}$.

FIG. 3. The box Ω_ϵ^+. On each successive cycle in the reverse time direction the trajectory intersects the box for a finite period of time coming progressively closer to the boundary of Ω (the arrows indicate the direction of the flow in the *forward* time sense).

As can be seen from Fig. 3, the set $\Sigma_1 + \Sigma_0$ encloses a box, Ω_ϵ^+, of area $\simeq \epsilon^2$ in Ω around $\mathbf{p}_+$. Another box, Ω_ϵ^-, can similarly be constructed around $\mathbf{p}_-$ by defining the sets $\Sigma_2 = \{(p,q) \in \Omega : p = \epsilon\ 1 + q \leq \epsilon\}$ and $\Sigma_3 = \{(p,q) : q = -1 + \epsilon,\ 0 < p \leq \epsilon\}$.

Let I be the time interval $[\tau_f, 0]$, then for a given trajectory $\psi_{\mathbf{p}_0}$ we may write $I = I_c \cup I_b$ where $I_c = \{\tau \in I : \psi_{\mathbf{p}_0}(\tau) \in \Omega_\epsilon^\pm\}$ and $I_b = \{\tau \in I : \psi_{\mathbf{p}_0}(\tau) \notin \Omega_\epsilon^\pm\}$. We thus have

$$\int_{\tau_f}^0 q^2 d\tau = \int_{I_c} q^2 d\tau + \int_{I_b} q^2 d\tau \qquad (26)$$

$$> \int_{I_c} q^2 d\tau. \qquad (27)$$

From the definition of I_c,

$$\int_{I_c} q^2 d\tau = \int_{I_c} (1 - O(\epsilon)) d\tau.$$

Fix n and let m be any number satisfying $n < m < 1$. Then for ϵ sufficiently small we have;

$$\int_{\tau_f}^0 q^2 d\tau > m \int_{I_c} d\tau. \qquad (28)$$

In order to complete the proof we must show that by choosing p_0 sufficiently small, the ratio of $\int_{I_c} d\tau$ to $\int_I d\tau$ may be made arbitrarily close to 1.

According to (15) p evolves according to the equation

$$\frac{d}{d\tau}\ln p = -\alpha q + q^2$$
$$< 1 + \alpha \tag{29}$$

since $|q| < 1$. Therefore if $\tau_2 < \tau_1$ we have

$$\ln \frac{p_1}{p_2} < (1+\alpha)(\tau_1 - \tau_2), \tag{30}$$

where $p_1 = p(\tau_1), p_2 = p(\tau_2)$. The parameter time $\Delta\tau$ needed for a point $\mathbf{p}_0 = (p_0, 1-\epsilon) \in \Sigma_0$ to flow through Ω_ϵ^+ and reach Σ_1 must therefore satisfy

$$|\Delta\tau| > \frac{1}{1+\alpha}\ln\left(\frac{\epsilon}{p_0}\right). \tag{31}$$

The modulus sign is necessary because we are tracing the trajectory backwards in time so $\Delta\tau$ is negative. As $p_0 \to 0$, $\Delta\tau \to -\infty$. Thus $\Delta\tau$ may be made arbitrarily large by choosing p_0 sufficiently small.

Since $\psi_{\mathbf{p}_0}$ is past asymptotic to the limit cycle γ_l, the intersection of the past orbit $O_{\mathbf{p}_0}^-$ with the line $q = 1-\epsilon$ (which contains Σ_0) must contain an infinite number of points in addition to $\mathbf{p}_0$ itself. If $\mathbf{p}_1$ is any such point then substituting $q = 1-\epsilon$ into (16) and using the monotonicity of H (18) we must have $p_1 < p_0$, where p_1 and p_0 are the p-coordinate values of $\mathbf{p}_1$ and $\mathbf{p}_0$ respectively. In other words $\psi_{\mathbf{p}_0}$ must intersect $\Sigma_{\epsilon 0}$ again after 1 complete cycle of $\partial\Omega$ and the point of intersection $\mathbf{p}_1$ must have a p-coordinate value p_1 which is smaller than p_0. It follows that $\psi_{\mathbf{p}_0}$ passes through the box Ω_ϵ^+ on each successive cycle and the time interval $\Delta\tau_n^+$ to traverse Ω_ϵ^+ on the nth cycle obeys the inequality;

$$|\Delta\tau_n^+| = \frac{1}{\alpha+1}\ln\left(\frac{\epsilon}{p_n}\right) \tag{32}$$

$$\geq \frac{1}{\alpha+1}\ln\left(\frac{\epsilon}{p_0}\right) \tag{33}$$

Similarly, defining $\Delta\tau_n^-$ to be the time taken for $\psi_{\mathbf{p}_0}$ to traverse the box Ω_ϵ^- on the nth cycle we find from (30) using an identical argument to that above that

$$|\Delta\tau_n^-| \geq \frac{1}{\alpha+1}\ln\left(\frac{\epsilon}{\tilde{p}_n}\right). \tag{34}$$

where $\tilde{p}_n$ is the p coordinate of the intersection of $\psi_{\mathbf{p}_0}$ with Σ_3 on the nth cycle (by the nth cycle I mean, precisely, one complete circuit from $(p_n - 1, 1-\epsilon) \in \Sigma_0$ to $(p_n, 1-\epsilon) \in \Sigma_0$). Recalling the definition of H and using the fact that q^2 takes the same value on Σ_0 and Σ_3 we have for any point on Σ_0 *or* Σ_3 that $H^2 = (2\epsilon-\epsilon^2)p^2 + O(\epsilon p^4)$. Since H is monotonic it follows that for ϵ chosen sufficiently small $\tilde{p}_n < p_0$. Therefore,

$$|\Delta\tau_n^-| \geq \frac{1}{\alpha+1}\ln\left(\frac{\epsilon}{p_0}\right). \tag{35}$$

Let us now consider the parameter time $\Delta\tau$ taken for a point on Σ_1 to reach Σ_2 under (15), close to the semicircular boundary $\partial\Omega_1$. That is, the time taken to flow backwards in time from the top box Ω_ϵ^+ to the bottom box Ω_ϵ^- (Fig. 4).
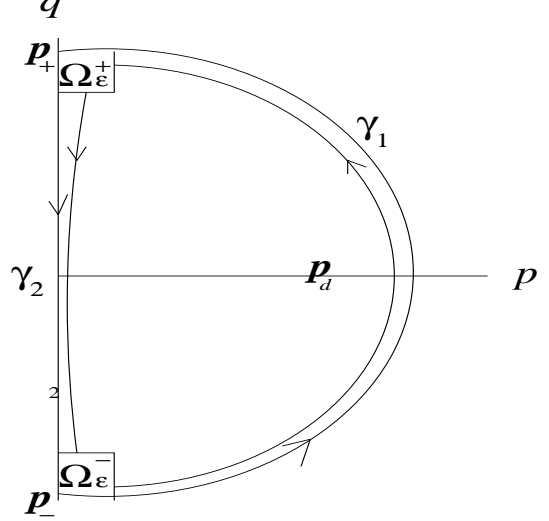
FIG. 4. The parts of the solution flowing between the boxes Ω_ϵ^+ and Ω_ϵ^- approach the restrictions of γ_1 and γ_2 to finite time intervals.

By continuity, $\Delta\tau$ approaches the (negative) parameter time interval which the asymptotic solution γ_1 takes to map the point $(\epsilon, \sqrt{1-\epsilon^2}) \in \Sigma_1$ to the point $(\epsilon, -\sqrt{1-\epsilon^2}) \in \Sigma_2$. This interval is finite.

Therefore, the time interval to go from Σ_1 to Σ_2 must approach a finite (negative) limit. Its modulus must therefore possess a finite upper bound τ_1.

Let $\Delta\tau_n^1$ be the time taken for ψ_{x_0} to go from Σ_1 to Σ_2 on the nth cycle. Then we have

$$|\Delta\tau_n^1| < \tau_1 \tag{36}$$

Similarly, if $\Delta\tau_n^2$ is the time taken for ψ_{x_0} to go backwards in time from Σ_3 to Σ_0 on the nth cycle we have

$$|\Delta\tau_n^2| < \tau_2 \tag{37}$$

For some finite number τ_2.

Now, Since ψ_{x_0} is incident on Σ_0 and hence, initially flows through Ω_ϵ^+, it follows from the inequalities (33), (35), (36) and (37) that

$$\frac{\int_{I_c} d\tau}{\int_{I_b} d\tau} > \frac{1}{\tau_\epsilon(\alpha+1)}\ln\left(\frac{\epsilon}{p_0}\right)$$

where $\tau_\epsilon = \max(\tau_1, \tau_2)$. Choose

$$p_0 = \epsilon \exp\left[-\frac{\tau_\epsilon(\alpha+1)}{(\frac{m}{n}-1)}\right],$$

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then we have

$$\int_I d\tau = \int_{I_c} d\tau \left(1 + \frac{\int_{I_m} d\tau}{\int_{I_c} d\tau}\right)$$

$$-\tau_f < \frac{m}{n} \int_{I_c} d\tau. \tag{38}$$

Combining (38), (28) and (25) gives

$$H(\tau_f) < H_0 e^{4n\tau_f}$$

which is the required result.□

We say a scalar field cosmology is non-trivial if there exists some space-time point for which $\dot{\phi}$ is non-zero.

Theorem 2 *If $(g_{\mu\nu}, \phi)$ is a non-trivial scalar field cosmology with potential (7) and if $g_{\mu\nu}$ is spatially flat and isotropic, then $g_{\mu\nu}$ possesses an initial space-time singularity.*

Proof:
By Lemma 1 there exists ϵ and p_m such that all trajectories of (15) incident on Σ_0 with $p_0 < p_m$ satisfy

$$H(\tau) < H_0 e^{2\tau}$$

for all $\tau < 0$. It will be sufficient to show that these trajectories reach the boundary in finite proper time. The proper time taken to reach the boundary is given by

$$\Delta t = \int_{-\infty}^0 x d\tau \tag{39}$$

$$< x_0 \int_{-\infty}^0 e^{\tau/2} d\tau \tag{40}$$

$$= 2x_0 \tag{41}$$

Since this is finite, we have the result.□

Theorem 3 *If $(g_{\mu\nu}, \phi)$ is a non-trivial scalar field cosmology with potential (7) and if $g_{\mu\nu}$ is spatially flat and isotropic, then particle horizons exist for all isotropic observers (observers whose worldlines are tangent to the timelike Killing vector field U^μ).*

Proof: A particle horizon exists for an isotropic observer at time t if the integral

$$l = \int_0^t \frac{1}{a} dt$$

exists and is finite. By the definition of τ;

$$a = e^{\frac{2}{3}\tau}$$

and

$$\frac{dt}{d\tau} = x$$

Therefore,

$$l = \int_{-\infty}^{\tau} e^{-\frac{2}{3}\tau} x d\tau.$$

By Lemma 1 there exists ϵ and p_m such that all trajectories of (15) incident on Σ_0 with $p_0 < p_m$ satisfy

$$H(\tau) < H_0 e^{\frac{5}{6}\tau} \tag{42}$$

for all $\tau < 0$. If l is finite at $\tau = 0$ it will be finite for all τ. It will therefore be sufficient to show that trajectories for which (42) holds possess a horizon at $\tau = 0$. Using (42) we have

$$l \le x_0 \int_{-\infty}^0 e^{\frac{\tau}{6}} d\tau \tag{43}$$

$$= 6x_0 \tag{44}$$

Since this is always finite we have the result.□

Note also that as $x_0 \to 0$, l must also approach 0 indicating that the horizon length shrinks to zero as $t \to 0$.

The physical meaning of the Lemma 1 becomes clearer when we recall that $\tau = \ln v$ and, by (17) and the definition of x $H = 9K^{-4}$. Lemma 1 may thus be interpreted as saying that for all $0 < n < 1$ there exists some $A > 0$ such that

$$K > Av^{-n}.$$

Recalling that $K = \dot{v}/v$ it thus follows that for all $m > 1$ there exists v_0 such that

$$v(t) > v_0 t^m \tag{45}$$

Thus, in the neighbourhood of the singularity the volume element expands faster than any power law with power greater than one. In order to avoid a particle horizon it must expand slower than t^3.

What about the case $m = 1$? Consider the function $e^{-4\tau} H$. Taking the derivative of this function with respect to τ, using (18)

$$\frac{d}{d\tau} e^{-4\tau} H = -4 e^{-4\tau} H(1 - q^2)$$

Thus for $\tau < 0$ we have

$$\ln \left(e^{-4\tau} H\right) = c + 4 \int_{\tau}^0 (1 - q^2) d\tau$$

where c is a constant. The integral on the right hand side tends to infinity as $\tau \to -\infty$ since each solution spends a finitely large amount of τ-time with, say, $q^2 < \epsilon$ on *each* cycle (of which there are an infinite number).

Thus,

$$\lim_{\tau \to -\infty} e^{-4\tau} H = \infty.$$

This translates to a corresponding limit for v and t, as above:

$$\lim_{t \to 0} t^{-1} v = 0. \tag{46}$$

Comparing this expression with (45) we see that v expands slower than t but faster than any power law t^p with $p > 1$. In this respect the behavior of the gravitational field near the singularity is quite subtle and unusual since it can not be adequately modeled by a power law. Note also that it is clearly not admissible to neglect the dynamical effect of the potential when considering the gravitational field near the singularity.

V. CONCLUSIONS

The above example indicates that non-trivial asymptotic behavior can emerge, even in very simple models, as a result of very steep self interaction potentials. It is of interest that both space-time singularity and particle horizons exist even though the strong energy condition is violated by typical solutions during all periods of their evolution, including asymptotically close to the singularity.

It is suggested by the author that the qualitative features exhibited above, including the oscillatory behavior and existence of a singularity and and particle horizons, are characteristic of FRW scalar field models which have very steep potential wells. Preliminary investigations of the steeper than exponential potential $V(\phi) = e^{\lambda \phi^2}$ and the "hard wall" potential $V(\phi) = 1/(\lambda - \phi^2)$ on the domain $\phi^2 < \lambda$ reveal no significant departure from the qualitative behavior of the exponential potential well.

The asymptotic oscillatory behavior of the scalar field is reminiscent of the oscillatory behavior of the components of the shear tensor displayed by the Belinskii, Khalatnikov, Lifschitz perfect fluid cosmologies [10–12]. This behavior is associated with the existence of dynamical chaos in the solution space of these cosmologies [13]. The apparent similarity suggests that it might be particularly interesting to investigate the dynamics of exponential potential well models in slightly more complicated space-times such as Bianchi type IX. It seems not inconcievable that the coupling of scalar field and shear degrees of freedom could increase the degree of mixing, thereby resulting in partical horizon avoidance, particularly since, unlike the BKL solutions, the expansion of scalar field cosmologies is not constrained by the strong energy condition.

Another interesting feature of the solutions is that there exists two natural time scales, namely the affine parameter time t and the period of oscillation of the scalar field, which we might choose to characterise by some coordinate η. The timelike geodesics are incomplete with respect to t but complete with respect η.

By convention we say that a space-time singularity exists if geodesics are incomplete with respect to their affine parameter since this describes a situation where a freely falling observer would reach the edge of space-time in finite proper time. However, the physical interpretation of t as the proper time is not really meaningful on a neighbourhood of a boundary point of space-time since no normal coordinates can be constructed at such a point (normal coordinates can be constructed at a point arbitrarily close to a singularity but may never be extended to the singularity itself).

On the other hand, η can be interpreted as the most natural time scale associated with the matter content of the universe (including all clocks and astronauts since scalar matter is the only matter entering into this model) and therefore might be a more meaningful measure of the time taken to reach the singularity. One could imagine that if a more realistic cosmological model could be shown to display a similar oscillatory behavior for its physical fields then one would be led to an interpretation whereby a singularity exists but can never be reached in a finite amount of time by any physical object.

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